

Representations of Vertex Operator Algebras $V_{L_2}^{S_4}$, $V_{L_2}^{A_5}$, and Their Quantum Dimensions

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Abstract

C_2 cofiniteness and rationality of $V_{L_2}^{S_4}$ are obtained, and irreducible $V_{L_2}^{S_4}$ -modules are classified. With the assumption of rationality and C_2 cofiniteness, irreducible $V_{L_2}^{A_5}$ -modules are determined. Also, quantum dimensions of these irreducible modules are calculated.

1 Introduction

Investigation of the vertex operator algebras $V_{L_2}^{S_4}$ and $V_{L_2}^{A_5}$ plays an important part in the classification of rational vertex operator algebras with $c = 1$. The vertex operator algebra $V_{L_2}^G$ consists of G invariants of the even lattice vertex operator algebra V_{L_2} .

In the literature of physics at character level, references [27] and [31] studied classification of rational vertex operator algebras with $c = 1$. These two references are based on two assumptions. one assumption is that the sum of the square norm of irreducible characters is invariant under the modular group. The second assumption is that each irreducible character is a modular function over a congruence subgroup. Reference [31] states that the character of a rational vertex operator algebra with $c = 1$ is the character of one of the vertex operators,

- (a) lattice vertex operator algebras V_L associated with positive definite even lattices L of rank one,
- (b) orbifold vertex operator algebras V_L^+ under the automorphism of V_L induced from the -1 isometry of L ,
- (c) $V_{L_2}^G$, where G is a finite subgroup of $SO(3)$ isomorphic to one of $\{A_4, S_4, A_5\}$.

Reference [9] indicates that the list above would not be correct if the effective central charge \tilde{c} , by Reference [20], and the central charge c were different. Reference [20] characterizes the vertex operator algebra V_L for any positive definite even lattice L by c , \tilde{c} , and the rank of the weight one subspace as a Lie algebra. References [9] [5] [10] [40] characterize the orbifold vertex operator algebra V_L^+ . References [11] [12] [13] characterize the vertex operator algebra $V_{L_2}^{A_4}$. Reference [11] states the rationality and the C_2 -cofiniteness of $V_{L_2}^{A_4}$. Since A_4 is of order two in S_4 , the C_2 -cofiniteness of $V_{L_2}^{A_4}$ implies the C_2 cofiniteness of $V_{L_2}^{S_4}$, by Reference [34].

Let V be a rational vertex operator algebra and G be a finite automorphism group of V . Orbifold theory conjecture indicates that V^G is rational and each irreducible V^G -module occurs in an irreducible g -twisted V -module for some $g \in G$. The second half of the conjecture is closed in Reference [6]. So, irreducible $V_{L_2}^{S_4}$ -modules are classified. Also, the vertex operator algebra $V_{L_2}^G$, where $G \in \{A_4, S_4, A_5\}$, is in the above list of rational vertex operator algebras. With the assumption of rationality and C_2 -cofiniteness, irreducible modules of $V_{L_2}^{A_5}$ can be determined.

Reference [14] defines and studies the quantum dimension and the global dimension for a V -module. Also, Reference [14] provides a quantum Galois theory $V^G \subseteq V$, by References [19] [28]. The results in Reference [14] are strengthened in Reference [6], and are important in determining irreducible modules of $V_{L_2}^{S_4}$ and $V_{L_2}^{A_5}$.

2 Preliminaries

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and g an automorphism of V of finite order T . Denote the decomposition of V into eigenspaces with respect to the action of g as

$$V = \bigoplus_{r \in \mathbb{Z}/T\mathbb{Z}} V^r,$$

where $V^r = \{v \in V | gv = e^{-2\pi ir/T} v\}$. Use r to denote both an integer between 0 and $T - 1$ and its residue class mod T in this situation.

Definition 2.1. A weak g -twisted V -module M is a vector space equipped with a linear map

$$\begin{aligned} Y_M : V &\rightarrow (\text{End } M)[[z^{1/T}, z^{-1/T}]] \\ v &\mapsto Y_M(v, z) = \sum_{n \in \frac{1}{T}\mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \text{End } M), \end{aligned}$$

which satisfies the following: for all $0 \leq r \leq T-1$, $u \in V^r$, $v \in V$, $w \in M$,

$$\begin{aligned} Y_M(u, z) &= \sum_{n \in \frac{r}{T} + \mathbb{Z}} u_n z^{-n-1}, \\ u_l w &= 0 \quad \text{for } l \gg 0, \\ Y_M(\mathbf{1}, z) &= Id_M, \end{aligned}$$

$$\begin{aligned} z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) Y_M(u, z_1) Y_M(v, z_2) - z_0^{-1} \delta \left(\frac{z_2 - z_1}{-z_0} \right) Y_M(v, z_2) Y_M(u, z_1) \\ = z_2^{-1} \left(\frac{z_1 - z_0}{z_2} \right)^{-r/T} \delta \left(\frac{z_1 - z_0}{z_2} \right) Y_M(Y(u, z_0)v, z_2), \end{aligned}$$

where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ and all binomial expressions (here and below) are to be expanded in nonnegative integral powers of the second variable.

Definition 2.2. A g -twisted V -module is a \mathbb{C} -graded weak g -twisted V -module M :

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$$

where $M_\lambda = \{w \in M | L(0)w = \lambda w\}$ and $L(0)$ is the component operator of $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}$. We also require that $\dim M_\lambda$ is finite and for fixed λ , $M_{\frac{n}{T} + \lambda} = 0$ for all small enough integers n .

If $w \in M_\lambda$, call λ as the weight of w and write $\lambda = \text{wt} w$.

Use \mathbb{Z}_+ to denote the set of nonnegative integers.

Definition 2.3. An admissible g -twisted V -module is a $\frac{1}{T}\mathbb{Z}_+$ -graded weak g -twisted V -module M :

$$M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M(n)$$

satisfying

$$v_m M(n) \subseteq M(n + \text{wt} v - m - 1)$$

for homogeneous $v \in V$, $m, n \in \frac{1}{T}\mathbb{Z}$.

By Reference [15], if $g = Id_V$ these g -twisted notations become the notions of weak, ordinary and admissible V -modules.

If $M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M(n)$ is an admissible g -twisted V -module, the contragredient module M' is defined as follows:

$$M' = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M(n)^*,$$

where $M(n)^* = \text{Hom}_{\mathbb{C}}(M(n), \mathbb{C})$. The vertex operator $Y_{M'}(a, z)$ is defined for $a \in V$ via

$$\langle Y_{M'}(a, z)f, u \rangle = \langle f, Y_M(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})u \rangle,$$

where $\langle f, w \rangle = f(w)$ is the natural pairing $M' \times M \rightarrow \mathbb{C}$. It follows from References [25] and [39] that $(M', Y_{M'})$ is an admissible g^{-1} -twisted V -module. Define the contragredient module M' for a g -twisted V -module M . In this case, M' is a g^{-1} -twisted V -module. Moreover, M is irreducible if and only if M' is irreducible.

Definition 2.4. *A vertex operator algebra V is called g -rational, if the admissible g -twisted module category is semisimple. V is called rational if V is 1-rational.*

There is another important concept called C_2 -cofiniteness, by Reference [41].

Definition 2.5. *A vertex operator algebra V is C_2 -cofinite if $V/C_2(V)$ is finite dimensional, where $C_2(V) = \langle v_{-2}u | v, u \in V \rangle$.*

The following results about g -rational vertex operator algebras are well-known, by References [16], [17].

Theorem 2.6. *If V is g -rational, the following statements hold.*

- (1) *Any irreducible admissible g -twisted V -module M is a g -twisted V -module. Moreover, there exists a number $\lambda \in \mathbb{C}$ such that $M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M_{\lambda+n}$ where $M_{\lambda} \neq 0$. The λ is called the conformal weight of M ;*
- (2) *There are only finitely many irreducible admissible g -twisted V -modules up to isomorphism.*
- (3) *If V is also C_2 -cofinite and g^i -rational for all $i \geq 0$ then the central charge c and the conformal weight λ of any irreducible g -twisted V -module M are rational numbers.*

Let V be a simple vertex operator algebra and G a finite and faithful group of automorphisms of V , and let $\text{Irr}(G)$ denote the set of simple characters χ of G . Now as $\mathbb{C}G$ -module, each homogeneous space V_n of V is of finite dimensional, and so there is a direct sum decomposition of V into graded subspaces

$$V = \bigoplus_{\chi \in \text{Irr}G} V^{\chi},$$

where V^{χ} is the subspace of V on which G acts according to the character χ . In other words, if M_{χ} is the simple $\mathbb{C}G$ -module affording χ , then V^{χ} is the M_{χ} -homogeneous subspace of V in the sense of group representation theory.

Theorem 2.1. *By Reference [20], suppose that V is a simple vertex operator algebra and that G is a finite and faithful solvable group of automorphisms of V . Then for $\chi \in \text{Irr}(G)$, each V^{χ} is a simple module for the G -graded vertex operator algebra $\mathbb{C}G \otimes V^G$ of the form*

$$V^{\chi} = M_{\chi} \otimes V_{\chi},$$

where M_{χ} is the simple $\mathbb{C}G$ -module affording χ and where V_{χ} is a simple V^G -module.

Theorem 2.2. *By reference [8]), let V be a simple vertex operator superalgebra and G a finite solvable subgroup of $\text{Aut}(V)$. Suppose that V^G is rational. Then V is g -rational for any $g \in G$.*

For a V -module M with grading $M = \oplus M_n$, define the formal character as

$$\text{ch}_q M = q^{-\frac{c}{24}} \sum \dim M_n q^n = \text{tr} q^{-\frac{c}{24} + L(0)}.$$

Denote the holomorphic function $\text{ch}_q M$ by $Z_M(\tau)$. Here and below, τ is in the upper half plane \mathbb{H} and $q = e^{2\pi i \tau}$.

Let V be a rational, C_2 -cofinite vertex operator algebra with central charge c . Zhu, in Reference [41], proved that the space

$$\left\langle q_1^{|a_1|} \cdots q_n^{|a_n|} \text{tr}_W Y(a_1, q_1) \cdots Y(a_n, q_n) q^{L(0)-c/24} : W \text{ irreducible } V\text{-modules} \right\rangle$$

is $SL_2(\mathbb{Z})$ -invariant with $a_i \in V_{|a_i|}$, where $q_j = q_{z_j} = e^{2\pi i z_j}$ and $|a_i|$ denotes the weight of a_j . The concept of g -twisted modules for a finite automorphism g was introduced in Reference [17] and the modular invariance of the space

$$\langle \text{tr}_M g^n q^{L(0)-c/24} : n \in \mathbb{Z}, M \text{ } g\text{-twisted modules} \rangle$$

was proved there.

Let M^0, \dots, M^d be the inequivalent irreducible V -modules where $M^0 \cong V$. Define

$$Z_i(u, v, \tau) = \text{tr}_{M^i} e^{2\pi i(v(0)) + (v,u)/2} q^{L(0)+u(0)+(u,u)/2-c/24}$$

for $u, v \in V_1$.

Theorem 2.3. *By Reference [33], let V be a rational, C_2 -cofinite vertex operator algebra of CFT type. Assume $u, v \in V_1$ such that u, v span an abelian Lie subalgebra of V_1 . Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$. Then $Z_i(u, v, q)$ converges to a holomorphic function in the upper half plane and*

$$Z_i(u, v, \gamma\tau) = \sum_{j=0}^d \gamma_{i,j} Z_j(au + bv, cu + dv, \tau),$$

where $\gamma_{i,j} \in \mathbb{C}$ are independent of the choice of u, v , $\gamma\tau = \frac{a\tau+b}{c\tau+d}$.

The convergence of these functions has been established in Reference [18].

The quantum dimensions of modules for vertex operator algebras are defined and their properties are discussed in Reference [14]. It is well known that $\text{qdim}_V M$ can be defined as the limit of $\frac{\text{Ch}_q M}{\text{Ch}_q V}$ as q goes to 1 from the left. The advantage of this definition is

that one can use the modular transformation property of the q -characters, by Reference [41], and Verlinde formula, by References [30] [37], to compute quantum dimensions and investigate their properties. The quantum dimensions for rational and C_2 -cofinite vertex operator algebras have some nice properties which enable us to determine fusion rules when the quantum dimensions can be calculated. The following are some properties of quantum dimensions, by Reference [14].

Definition 2.4. Let V be a vertex operator algebra and M a g -twisted V -module such that $Z_V(\tau)$ and $Z_M(\tau)$ exists. The quantum dimension of M over V is defined as

$$\text{qdim}_V M = \lim_{y \rightarrow 0} \frac{Z_M(iy)}{Z_V(iy)},$$

where y is real and positive.

By Reference [17], there is a natural action of $\text{Aut}(V)$ on twisted modules. Let g, h be two automorphisms of V with g of finite order. If M, Y_g is a weak g -twisted V -module, there is a weak hgh^{-1} twisted V modules $(M \circ h), Y_{hgh^{-1}}$ where $M \circ h \cong M$ as vector spaces and

$$Y_{hgh^{-1}}(v, z) = Y_g(h^{-1}v, z)$$

for $v \in V$. this defines a left action of $\text{Aut}(V)$ on weak twisted V -modules and on isomorphism classes of weak twisted V -modules and on isomorphism classes of weak twisted V -modules. Symbolically, write

$$h \circ (M, Y_g) = (M \circ h, Y_{hgh^{-1}}) = h \circ M.$$

Sometimes abuse notation slightly by identifying M, Y_g with the isomorphism class that it defines.

If g, h commute, obviously h acts on the g -twisted modules as above. Set $\mathcal{M}(g)$ to be the equivalence classes of irreducible g -twisted V -modules and $\mathcal{M}(g, h) = \{M \in \mathcal{M}(g) | h \circ M \cong M\}$. Then, for any $M \in \mathcal{M}(g, h)$, there is a g -twisted V -module isomorphism

$$\varphi(h) : h \circ M \rightarrow M.$$

The linear map $\varphi(h)$ is unique up to a nonzero scalar.

Definition 2.5. By Reference [14], define the *global dimension* of V as

$$\text{glob}(V) = \sum_{i=0}^d (\text{qdim}_V M^i)^2.$$

3 Basic results of irreducible modules

In the rest of the paper, assume the following if not specified.

- (V1) $V = \oplus_{n \geq 0} V_n$ is a simple vertex operator algebra of CFT type,
- (V2) G is a finite automorphism group of V and V^G is a vertex operator algebra of CFT type,
- (V3) V^G is C_2 -cofinite and rational,
- (V4) The conformal weight of any irreducible V^G -module N is nonnegative and is zero if and only if $N = V^G$.

Let V be a vertex operator algebra, (W, Y) an irreducible V -module, and g an automorphism of V . Define a linear map

$$Y^\sigma : V \rightarrow (\text{End } W)[[z, z^{-1}]]$$

by

$$Y^\sigma(u, z)w = Y(\sigma^{-1}(u), z)w,$$

where $u \in V$, and $w \in W$. Reference [16] shows that (W, Y^σ) is an irreducible V -module. Denote (W, Y^σ) by W^σ .

Definition 3.1. By Reference [16], a V -module W is *g stable* if $W \cong W^\sigma$.

Theorem 3.2. *By Reference [16], the cardinalities $|\mathcal{M}(g, h)|$ and $|\mathcal{M}(g^a h^c, g^b h^d)|$ are equal for any $(g, h) \in P(V)$ and $\gamma \in \Gamma$. In particular, the number of irreducible g -twisted V -module is exactly the number of irreducible V -modules which are g -stable.*

The next two lemmas are from Reference [19], and provide a practical way to construct irreducible V^G modules.

Lemma 3.3. *Let V be a vertex operator algebra with an automorphism g of order T . Let $M = \sum_{n \in \frac{1}{T}\mathbb{Z}_+} M(n)$ be an irreducible g -twisted admissible V -Module. Then $M^i = \bigoplus_{n \in \frac{i}{T} + \mathbb{Z}} M(n)$ is an irreducible $V^{(g)}$ -module for $i = 0, \dots, T-1$.*

Lemma 3.4. *Let V be a simple vertex operator algebra, g an automorphism of V of prime order p , and M an irreducible V -module such that $g \circ M$ is not isomorphic to M as V -modules. Then, M is an irreducible $V^{(g)}$ -module.*

The next two theorems are from Reference [6], and are the key theorems in determining irreducible modules of a rational, C_2 cofinite vertex operator algebra.

Theorem 3.5. *Let $g, h \in G$, M an irreducible g -twisted V -module, N an irreducible h -twisted V -module. Also assume that M, N are not in the same orbit of \mathcal{S} under the action of G . Then*

- 1) *Each M_λ for $\lambda \in \Lambda_{G_M, \alpha_M}$ is an irreducible V^G -module.*
- 2) *For any $\lambda \in \Lambda_{G_M, \alpha_M}$ and $\mu \in \Lambda_{G_N, \alpha_N}$, the irreducible V^G -modules M_λ and N_μ are inequivalent.*

Theorem 3.6. *Any irreducible V^G -module is isomorphic to an irreducible V^G -submodule M_λ for some irreducible g -twisted V -module M and some $\lambda \in \Lambda_{G_M, \alpha_M}$.*

Remark 3.7. Theorem 3.6 shows that there are two types of irreducible V^G modules.

- An irreducible V^G module M is of *type one* if M occurs in the decomposition of irreducible V modules, as V^G modules.
- An irreducible V^G module M is of *type two* if M does not occur in the decomposition of irreducible V modules, as V^G modules. That is, M occurs in a g twisted V^G module for some $g \in G$ and $g \neq 1$.

Theorem 3.8. *Let V be a rational vertex operator algebra, and G an automorphism of V . Let H be a subgroup of G such that $C_G(h) = H$ for each $h \in H$. Assume that V^H is rational. (Note that V^G is not necessarily to be rational). Then, an irreducible V^H module of type two is an irreducible V^G module of type two.*

Proof. Let M be an irreducible V^H module of type two. Then it occurs in some g -twisted V module N with $g \in H$. Let G_N be the subgroup of G consisting of $h \in G$ such that $N \circ h \cong N$. Note that $G_N \subset C_G h = H$. So, G_N is the subgroup of H consisting of $h \in H$ such that $N \circ h \cong N$. Let (G_N, α_N) be a projective representation of G_N on N . Theorem 3.5 shows that

$$N = \bigoplus_{\lambda \in \text{irr}(G_N, \alpha_N)} W_\lambda \otimes N_\lambda. \quad (3.1)$$

Then $M = N_\lambda$ for some λ . Since each N_λ is an irreducible V^G module, M is an irreducible V^G module. \square

Theorem 3.9. *By Reference [6], one has*

$$\text{qdim}_{V^G} M = |G| \text{qdim}_V M.$$

for any irreducible g -twisted V -module M .

Theorem 3.10. *By Reference [6], one has the following relation,*

$$\text{glob}(V^G) = |G|^2 \text{glob}(V).$$

Theorem 3.11. *By Reference [13], let V be a rational, C_2 -cofinite vertex operator algebra with central charge c and M^0, \dots, M^d be the inequivalent irreducible V -modules with $M^0 \cong V$. Then $\text{qdim} M^i = \text{qdim}(M^i)^\sigma$.*

Theorem 3.12. *Let V be a vertex operator algebra, and G an automorphism group of V . Let H_1 and H_2 be two subgroups of G such that $g^{-1}H_1g = H_2$ for some $g \in G$, that is, H_1 and H_2 are conjugate under G . Let M_1 be an irreducible V^{H_1} module. Then, there exists an irreducible V^{H_2} module, M_2 such that as V_G modules, $M_1 \cong M_2$.*

Proof. Let $h_2 \in H_2$, $v \in V^{H_1}$. Then, there exists $h_1 \in H_1$ such that $g^{-1}h_1g = h_2$, that is $g^{-1}h_1 = h_2g^{-1}$

$$\begin{aligned} h_2(g^{-1}v) &= g^{-1}h_1v \\ &= g^{-1}v. \end{aligned}$$

This shows that $g^{-1}v \in V^{H_2}$. Hence, $g^{-1}V^{H_1} \subseteq V^{H_2}$. Likewise, $gV_{H_2} \subseteq V^{H_1}$. Thus,

$$\begin{aligned} V^{H_2} &= g^{-1}(gV^{H_2}) \\ &\subseteq g^{-1}V^{H_1} \\ &\subseteq V^{H_2}. \end{aligned}$$

So, $g^{-1}V^{H_1} = V^{H_2}$.

Let $M_1 \circ g \cong M_1$ as vector spaces. Consider $(M_1 \circ g, Y_{M_1 \circ g})$ as a V^{H_2} module. For $v \in V^{H_2}$, define

$$Y_{M_1 \circ g}(v, z) := Y_{M_1}(gv, z).$$

The fact $g^{-1}V^{H_1} = V^{H_2}$ shows that $Y_{M_1 \circ g}$ is well defined, because g is an automorphism of V . Write

$$(M_1, Y_{M_1}) \circ g = (M_1 \circ g, Y_{M_1 \circ g}) = M_1 \circ g.$$

Let N be a submodule of $M_1 \circ g$, as V^{H_2} modules. Then, $N \circ g^{-1}$ is a submodule of M_1 , as V^{H_1} modules. The fact that M_1 is an irreducible V^{H_1} module shows that $N \circ g^{-1}$ is trivial in M_1 . Thus, $N = ((N \circ g^{-1}) \circ g)$ is trivial in $M_1 \circ h$. So, $M_1 \circ g$ is an irreducible V^{H_2} module.

Let $w \in V^G$. Then, $gw = w$, and hence $Y_G(w, z) = Y_G(gw, z)$. That is, $M_1 \circ g \cong M_1$, as V_G modules. \square

Remark 3.13. Denote by $\mathcal{M}(H)$ the set of all irreducible modules of V^H . Let $\{H_i | i \in I\}$ be a conjugacy class of an automorphism group G . Theorem 3.12 shows that $\mathcal{M}(H_i)$, for $i \in I$, are the same, as V^G modules.

Recall the definition of Schur covering group. The following materials are based on References [38] [29] [36].

Definition 3.14. A group homomorphism from D to G is said to be a Schur cover of the finite group G if:

- 1. the kernel is contained both in the center and the derived subgroup of D , and
- 2. amongst all such homomorphism, this D has maximal size.

A group D is a Schur covering group for G if there is a Schur cover from D to G .

The Schur covers of the symmetric and alternating groups were classified.

Theorem 3.15. *The symmetric group of degree $n \geq 4$ has two isomorphic classes of Schur covers, both of order $2 \cdot n!$. Then alternating group of degree n has one isomorphic class of Schur covers, which has order $n!$ except when n is 6 or 7.*

Theorem 3.16. *For $n = 4$, the Schur cover of the alternating group A_4 is given by $SL(2, 3) \rightarrow PSL(2, 3) \cong A_4$. The Schur covers of the symmetric group S_4 are $GL(2, 3)$ and binary octahedral group. For $n = 5$, the Schur cover of the alternating group A_5 is given by $SL(2, 5) \rightarrow PSL(2, 5) \cong A_5$.*

There is a construction of Schur covering group for A_4 . Let $\pi : SU(2) \rightarrow SO(3)$ be the natural homomorphism. There is an injective homomorphism $i : A_4 \rightarrow SO(3)$ such that $\pi^{-1}(i(A_4))$ is a Schur covering group of A_4 .

Since A_4 has only one 3-dimensional irreducible representation, for any injective homomorphism $i : A_4 \rightarrow SO(3)$, $\pi^{-1}(i(A_4))$ is a Schur covering group of A_4 .

Lemma 3.17. *Let G be a finite group such that $A_4 \subset G$. Let $\pi : SU(2) \rightarrow SO(3)$ be the canonical homomorphism. Let $\phi : G \rightarrow SO(3)$ be an injective homomorphism. Then the kernel of ϕ is contained both in the center and the derived subgroup of $\pi^{-1}(G)$.*

Proof. Note that $\ker(\pi) = \{\text{Id}, -\text{Id}\} \subset SU(2)$ which is the center of $SU(2)$. So $\ker(\pi)$ is contained in the center of $\pi^{-1}(\phi(G))$. Since $A_4 \subset G$, $\pi^{-1}(\phi(A_4)) \subset \pi^{-1}(\phi(G))$. Note that $\pi^{-1}(\phi(A_4))$ is a Schur covering group of A_4 . So $\ker(\pi)$ is contained in the derived subgroup of $\pi^{-1}(\phi(A_4))$ and is also contained in the derived subgroup of $\pi^{-1}(\phi(G))$. \square

Remark 3.18. Let ϕ be an injective homomorphism from A_5 to $SO(3)$. Note that the degree of Schur covering group of A_5 is 120 which is the same as the degree of the group $\pi^{-1}(\phi(A_5))$. By the definition of Schur covering group, $\pi^{-1}(\phi(A_5))$ is the unique Schur covering group of A_5 .

4 The vertex operator algebra $V_{L_2}^{A_4}$

Let $L = \mathbb{Z}\alpha$ be a positive definite even lattice of rank one. That is $(\alpha, \alpha) = 2k$ for some positive integer k . $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$ and $\hat{\mathfrak{h}}_{\mathbb{Z}}$ the corresponding Heisenberg algebra; the bilinear form on L or \mathfrak{h} is denoted $\langle \cdot, \cdot \rangle$. Denote by $M(1)$ the associated irreducible module for $\hat{\mathfrak{h}}_{\mathbb{Z}}$ such that the canonical central element of $\hat{\mathfrak{h}}_{\mathbb{Z}}$ acts as 1. Let $\mathbb{C}[L]$ be the group algebra of L with a basis e^α for $\alpha \in L$. Let $\beta \in \mathfrak{h}$ such that $\langle \beta, \beta \rangle = 1$. It was proved in References [4] [26] that there is a linear map

$$V_L \rightarrow (\text{End} V_L) [[z, z^{-1}]]$$

$$v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \text{End} V_L)$$

such that $V_L = (V_L, Y, \mathbf{1}, \omega)$ is a simple vertex operator algebra where $\mathbf{1} = 1 \otimes e^0$. The dual lattice L° of L is

$$L^\circ = \{\lambda \in \mathfrak{h} \mid (\alpha, \lambda) \in \mathbb{Z}\} = \frac{1}{2k}L.$$

Then $L^\circ = \cup_{i=-k+1}^k (L + \lambda_i)$ is the coset decomposition with $\lambda_i = \frac{i}{2k}\alpha$. Set $V_{L+\lambda_i} = M(1) \otimes \mathbb{C}[L + \lambda_i]$. Then $V_{L+\lambda_i}$ for $i = -k+1, \dots, k$ are the irreducible modules for V_L . Let θ be an automorphism of \hat{L} such that $\theta(\alpha) = -\alpha$. We define an automorphism of V_L , denote again by θ , such that

$$\theta(u \otimes e^\alpha) = \theta(u) \otimes e^{-\alpha} \text{ for } u \in M(1) \text{ and } \alpha \in \hat{L}.$$

Here the action of θ on $M(1)$ is given by

$$\theta(\beta(n_1) \cdots \beta(n_k)) = (-1)^k \beta(n_1) \cdots \beta(n_k).$$

The θ -invariant V_L^+ of V_L form a simple vertex operator subalgebra and the (-1) -eigenspace V_L^- is an irreducible V_L^+ -module. Clearly $V_L = V_L^+ \oplus V_L^-$.

Let χ_s be a character of $L/2L$ such that $\chi_s(\alpha) = (-1)^s$ for $s = 0, 1$ and $T_{\chi_s} = \mathbb{C}$ the irreducible $L/2L$ -module with character χ_s . It is well known that $V_L^{T_s} = M(1)(\theta) \otimes T_{\chi_s}$ is an irreducible θ -twisted V_L -module. By References [26] [24], denote the ± 1 -eigenspaces of $V_L^{T_s}$ under θ by $(V_L^{T_s})^\pm$.

Theorem 4.1. *By Reference [22], any irreducible V_L^+ -module is isomorphic to one of the following modules,*

$$V_L^\pm, V_{\lambda_i+L} (i \neq k), V_{\lambda_k+L}, (V_L^{T_s})^\pm.$$

Theorem 4.2. *The quantum dimensions for all irreducible V_L^+ -modules over V_L^+ are given by the following tables.*

	V_L^+	V_L^-	$V_{L+\frac{r}{2k}\alpha} (1 \leq r \leq k-1)$	$V_{L+\frac{\alpha}{2}+}$	$V_{L+\frac{\alpha}{2}-}$
ω	0	1	$\frac{r^2}{4k}$	$\frac{k}{4}$	$\frac{k}{4}$
qdim	1	1	2	1	1

	$V_L^{T_1,+}$	$V_L^{T_1,-}$	$V_L^{T_2,+}$	$V_L^{T_2,-}$
ω	$\frac{1}{16}$	$\frac{9}{16}$	$\frac{1}{16}$	$\frac{9}{16}$
qdim	k	k	k	k

Proof. Reference [22] shows the action of ω on the first level of each irreducible module. The definition of $V_L^{T_i,\pm}$, for $i = 1, 2$, indicates

$$\text{qdim}_{V_L^+} V_L^{T_i,\pm} = \text{qdim}_{V_L^+} V_{\mathbb{Z}\alpha}.$$

Hence, the quantum dimensions listed are obtained by Theorem 3.9 and Theorem 3.10.

□

Let L_2 be the root lattice of type A_1 and A_4 the alternating group which is a subgroup of the automorphism group of lattice vertex operator algebra V_{L_2} . Motivated by the classification of rational vertex operator algebras with $c = 1$, $V_{L_2}^{A_4}$ was studied in Reference [11]. The C_2 -cofiniteness and rationality of $V_{L_2}^{A_4}$ are obtained, and the irreducible modules are classified. They first realize $V_{\mathbb{Z}\alpha}^G$ as $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$ where $\langle\beta, \beta\rangle = 8$ and σ is an automorphism of $sl(2, \mathbb{C})$ of order 3, since $V_{\mathbb{Z}\beta}^+$ is well understood, by References [1] [3] [2] [22] [23] [21].

Let $L_2 = \mathbb{Z}\alpha$ be the rank one positive-definite even lattice such that $(\alpha, \alpha) = 2$ and V_{L_2} the associated simple rational vertex operator algebra. Then $(V_{L_2})_1 \cong sl_2(\mathbb{C})$ and $(V_{L_2})_1$ has an orthonormal basis:

$$x^1 = \frac{1}{\sqrt{2}}\alpha(-1)\mathbf{1}, \quad x^2 = \frac{1}{\sqrt{2}}(e^\alpha + e^{-\alpha}), \quad x^3 = \frac{i}{\sqrt{2}}(e^\alpha - e^{-\alpha}).$$

Let $\tau_i \in \text{Aut}(V_{L_2})$, $i = 1, 2, 3$ be such that

$$\begin{aligned} \tau_1(x^1, x^2, x^3) &= (x^1, x^2, x^3) \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}, \\ \tau_2(x^1, x^2, x^3) &= (x^1, x^2, x^3) \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix}, \\ \tau_3(x^1, x^2, x^3) &= (x^1, x^2, x^3) \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}. \end{aligned}$$

Let $\sigma \in \text{Aut}(V_{L_2})$ be such that

$$\sigma(x^1, x^2, x^3) = (x^1, x^2, x^3) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}.$$

Then σ and τ_i , $i = 1, 2, 3$, generate a finite subgroup of $\text{Aut}(V_{L_2})$ isomorphic to the alternating group A_4 . Denote this group by A_4 . The subgroup K generated by τ_i , $i = 1, 2, 3$, is a normal subgroup of A_4 of order 4. Let $\beta = 2\alpha$.

Lemma 4.3. *By Reference [7], $V_{L_2}^K \cong V_{\mathbb{Z}\beta}^+$.*

Thus, $V_{L_2}^{A_4} = (V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$.

Let W^{1,T_1} and W^{2,T_1} be the only two irreducible σ -twisted modules of $V_{\mathbb{Z}\beta}^+$ and W^{1,T_2} , W^{2,T_2} be the only two irreducible σ^2 -twisted modules of $V_{\mathbb{Z}\beta}^+$. Denote irreducible $V_{\mathbb{Z}\beta}^+$ -submodules

of W^{i,T_j} by $W^{i,T_j,k}$, $i, j = 1, 2$; $k = 1, 2, 3$ which are irreducible $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$ -modules. Then there are exactly 21 irreducible modules of $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$ which could be listed as following, by Reference [11].

$$\{(V_{\mathbb{Z}\beta}^+)^m, V_{\mathbb{Z}\beta}^-, V_{\mathbb{Z}\beta+\frac{1}{8}\beta}, V_{\mathbb{Z}\beta+\frac{3}{8}\beta}, W^{i,T_j,k}, (V_{\mathbb{Z}\beta+\frac{1}{4}\beta})^n \\ |m, n = 0, 1, 2; i, j = 1, 2; k = 1, 2, 3\}.$$

Lemma 4.4. *By Reference [13], the group $SO(3)$ is the connected compact subgroup of $\text{Aut}(V_{L_2})$, whose discrete subgroup are the cyclic group Z_n , the dihedral group D_n , A_4 , S_4 and A_5 . Also, the vertex operator algebra $V_{L_2}^{Z_n} \cong V_{\mathbb{Z}n\alpha}$, and $V_{L_2}^{D_n} \cong V_{\mathbb{Z}n\alpha}^+$.*

Lemma 4.5. *By Reference [19], let g be an automorphism of the vertex operator algebra V_{L_2} and of order T . Let $\nu = T\alpha$. Then, eigenvalues of the g action on $V_{\mathbb{Z}}$ show that*

$$V_{L_2} \cong \bigoplus_{i \in \mathbb{Z}, 0 \leq i \leq T-1} V_{\mathbb{Z}\nu + \frac{i}{T}\nu},$$

and

$$V_{L_2+\frac{1}{2}\alpha} \cong \bigoplus_{i \in \mathbb{Z}, 0 \leq i \leq T-1} V_{\mathbb{Z}\nu - \frac{\nu}{2T} + \frac{i}{T}\nu}.$$

Remark 4.6. Let g be an automorphism of the vertex operator algebra V_{L_2} and of order T . Let $\xi = T\alpha$. Notice that $V_{L_2} \cong V_{L_2}^g$ and $V_{L_2+\frac{\alpha}{2}} \cong V_{L_2+\frac{\alpha}{2}}^g$, that is, V_{L_2} and $V_{L_2+\frac{\alpha}{2}}$ are g stable. Likewise, V_{L_2} and $V_{L_2+\alpha/2}$ are g^i stable, where $i \in \mathbb{Z}$, $0 \leq i \leq T-1$. Theorem 3.2 shows that there are exactly two irreducible g^i twisted modules, where $i \in \mathbb{Z}$, $0 \leq i \leq T-1$. Lemma 4.5 and Theorem 3.5 show that $V_{\mathbb{Z}\xi+\frac{r}{T}\xi}$ for $r = 0 \pmod{T}$ are irreducible $V_{L_2}^{\langle g \rangle}$ modules occurring in g^0 twisted V_{L_2} modules. Hence, $V_{\mathbb{Z}\xi+\frac{r}{T}\xi}$ for $r \neq 0 \pmod{T}$ are irreducible $V_{L_2}^{\langle g \rangle}$ modules occurring in g^r twisted V_{L_2} modules, where $r \in \mathbb{Z}$, $1 \leq r \leq T-1$.

Theorem 4.7. *Irreducible modules of $V_{L_2}^{A_4}$ are*

$$(V_{\mathbb{Z}\beta}^+)^0, (V_{\mathbb{Z}\beta}^+)^1, (V_{\mathbb{Z}\beta}^+)^2, \\ V_{\mathbb{Z}\beta}^-, V_{\mathbb{Z}\beta+\frac{1}{8}\beta}, V_{\mathbb{Z}\beta+\frac{3}{8}\beta}, \\ V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^0, V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^1, V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^2, \\ V_{\mathbb{Z}\gamma \pm \frac{r}{18}\gamma}, \text{ for } r \in \mathbb{Z}, 1 \leq r \leq 8, \text{ and } r \neq 0 \pmod{3}.$$

Reference [11] gives twenty one irreducible modules of $V_{L_2}^{A_4}$ by analyzing the action of σ on irreducible modules of $V_{\mathbb{Z}\beta}^+$, and by constructing all irreducible σ^i , for $i = 1, 2$, twisted $V_{\mathbb{Z}\beta}^+$ modules. Reference [11] shows that twelve of the twenty one irreducible modules come from irreducible σ^i , for $i = 1, 2$, twisted $V_{\mathbb{Z}\beta}^+$ modules. That is, they are of type two. These twelve irreducible modules are found from irreducible $V_{\mathbb{Z}\gamma}$ modules, by Reference

[13], without specifying twisted modules.

Let $H = \langle \sigma \rangle$. Then, $H \subseteq A_4$ satisfies the assumption in Theorem 3.8. Remark 4.6 shows that $V_{\mathbb{Z}\gamma \pm \frac{r}{18}}$, for $r \in \mathbb{Z}$, $1 \leq r \leq 8$, and $r \not\equiv 0 \pmod{3}$ are twelve irreducible $V_{L_2}^{(\sigma)}$ modules of type two. Theorem 3.8 indicates that the twelve irreducible $V_{L_2}^{(\sigma)}$ modules of type two are twelve irreducible $V_{L_2}^{A_4}$ modules of type two, probably isomorphic under $V_{L_2}^{A_4}$. Also, theorem 3.6 indicates that the twelve irreducible $V_{L_2}^{(\sigma)}$ modules of type two exhaust irreducible $V_{L_2}^{A_4}$ modules of type two. Thus, Theorems 3.9 and 3.10 show that the twelve irreducible $V_{L_2}^{(\sigma)}$ modules of type two are exactly the twelve irreducible $V_{L_2}^{A_4}$ modules of type two. That is, they are nonisomorphic under $V_{L_2}^{A_4}$. The same idea will be used to find irreducible modules of $V_{L_2}^{S_4}$.

5 Irreducible modules of $V_{L_2}^{S_4}$

Let $\zeta = 4\alpha$, and ρ an automorphism of V_{L_2} such that $\rho(x^1) = -x^1$, $\rho(x^2) = x^3$, and $\rho(x^3) = x^2$. That is,

$$\rho(x^1, x^2, x^3) = (x^1, x^2, x^3) \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

Then, $\rho \in S_4 \setminus A_4$, and $V_{L_2}^{S_4} \cong (V_{L_2}^{A_4})^{(\rho)}$. Definition of automorphism on vertex operator algebra shows that $\rho(\omega) = \omega$. Observe the table in the next theorem, by Reference [13]

Theorem 5.1. *By Reference [35], let V be a C_2 -cofinite simple vertex operator algebra of CFT-type and $\sigma \in \text{Aut}(V)$ of finite order p . Then a fixed point vertex operator subalgebra V^σ is also C_2 -cofinite.*

Remark 5.2. The preceding theorem shows that the vertex operator algebra $V_{L_2}^{S_4} \cong (V_{L_2}^{A_4})^{(\rho)}$ is C_2 cofinite. Reference [11] shows that $V_{L_2}^{A_4}$ is rational. Also, the simplicity of $V_{L_2}^{A_4}$ shows that $V_{L_2}^{A_4} \cong (V_{L_2}^{A_4})'$. Thus, Condition I in Reference [32] is satisfied. Hence, Corollary 28 in Reference [32] indicates that $V_{L_2}^{S_4} \cong (V_{L_2}^{A_4})^{(\rho)}$ is rational.

Theorem 5.3. *The quantum dimensions for all irreducible $(V_{\mathbb{Z}\beta}^+)^{(\sigma)}$ -modules are given by the following tables 1 – 4.*

	$(V_{\mathbb{Z}\beta}^+)^0$	$(V_{\mathbb{Z}\beta}^+)^1$	$(V_{\mathbb{Z}\beta}^+)^2$	$V_{\mathbb{Z}\beta}^-$	$V_{\mathbb{Z}\beta + \frac{1}{8}\beta}$	$V_{\mathbb{Z}\beta + \frac{3}{8}\beta}$
ω	0	4	4	1	$\frac{1}{16}$	$\frac{9}{16}$
$qdim$	1	1	1	3	6	6

	$W^{1,T_1,0}$	$W^{1,T_1,1}$	$W^{1,T_1,2}$	$W^{2,T_1,0}$	$W^{2,T_1,1}$	$W^{2,T_1,2}$
ω	$\frac{1}{36}$	$\frac{25}{36}$	$\frac{49}{36}$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{16}{9}$
$qdim$	4	4	4	4	4	4

	$W^{1,T_2,0}$	$W^{1,T_2,1}$	$W^{1,T_2,2}$	$W^{2,T_2,0}$	$W^{2,T_2,1}$	$W^{2,T_2,2}$
ω	$\frac{1}{36}$	$\frac{25}{36}$	$\frac{49}{36}$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{16}{9}$
$qdim$	$\frac{4}{4}$	$\frac{4}{4}$	$\frac{4}{4}$	$\frac{4}{4}$	$\frac{4}{4}$	$\frac{4}{4}$

	$\left(V_{\mathbb{Z}\beta+\frac{1}{4}\beta}\right)^0$	$\left(V_{\mathbb{Z}\beta+\frac{1}{4}\beta}\right)^1$	$\left(V_{\mathbb{Z}\beta+\frac{1}{4}\beta}\right)^2$
ω	$\frac{1}{4}$	$\frac{9}{4}$	$\frac{9}{4}$
$qdim$	$\frac{2}{2}$	$\frac{2}{2}$	$\frac{2}{2}$

The fact that $\rho(\omega) = \omega$ shows that the irreducible modules with distinguished ω actions are ρ stable. That is, $(Z_{\mathbb{Z}\beta}^+)^0$, $V_{\mathbb{Z}\beta}^-$, $V_{\mathbb{Z}\beta+\frac{1}{8}\beta}$, $V_{\mathbb{Z}\beta+\frac{3}{8}\beta}$, $(V_{\mathbb{Z}\beta+\frac{1}{4}\beta})^0$, are ρ stable. For a ρ -invariant subspace W of $V_{\mathfrak{h}} = M(1) \otimes \mathbb{C}[\mathfrak{h}]$, abuse the notation W^\pm for the ± 1 -eigenspaces of W under ρ .

Lemma 5.4. *The following 10 spaces are irreducible $V_{L_2}^{S_4}$ modules,*

$$\begin{aligned}
& ((V_{\mathbb{Z}\beta}^+)^0)^+, ((V_{\mathbb{Z}\beta}^+)^0)^-, \\
& (V_{\mathbb{Z}\beta}^-)^+, (V_{\mathbb{Z}\beta}^-)^-, \\
& (V_{\mathbb{Z}\beta+\frac{1}{8}\beta})^+, (V_{\mathbb{Z}\beta+\frac{1}{8}\beta})^-, \\
& (V_{\mathbb{Z}\beta+\frac{3}{8}\beta})^+, (V_{\mathbb{Z}\beta+\frac{3}{8}\beta})^-, \\
& (V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^0)^+, ((V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^0)^-).
\end{aligned}$$

Proof. The order of ρ is 2. So, directly employ Theorem 3.5. \square

Proposition 5.5. *By Reference [13], as irreducible $V_{L_2}^{A_4}$ modules,*

$$\begin{aligned}
V_{\mathbb{Z}\gamma+\frac{1}{18}\gamma} &\cong W^{1,T_1,0}, \quad V_{\mathbb{Z}\gamma-\frac{1}{18}\gamma} \cong W^{1,T_2,0}, \\
V_{\mathbb{Z}\gamma+\frac{2}{18}\gamma} &\cong W^{2,T_2,0}, \quad V_{\mathbb{Z}\gamma-\frac{2}{18}\gamma} \cong W^{2,T_1,0}, \\
V_{\mathbb{Z}\gamma+\frac{4}{18}\gamma} &\cong W^{2,T_1,1}, \quad V_{\mathbb{Z}\gamma-\frac{4}{18}\gamma} \cong W^{2,T_2,1}, \\
V_{\mathbb{Z}\gamma+\frac{5}{18}\gamma} &\cong W^{1,T_2,1}, \quad V_{\mathbb{Z}\gamma-\frac{5}{18}\gamma} \cong W^{1,T_1,1}, \\
V_{\mathbb{Z}\gamma+\frac{7}{18}\gamma} &\cong W^{1,T_1,2}, \quad V_{\mathbb{Z}\gamma-\frac{7}{18}\gamma} \cong W^{1,T_2,2}, \\
V_{\mathbb{Z}\gamma+\frac{8}{18}\gamma} &\cong W^{2,T_2,2}, \quad V_{\mathbb{Z}\gamma-\frac{8}{18}\gamma} \cong W^{2,T_1,2},
\end{aligned}$$

Let $H = \langle \sigma \rangle$. Then, $H \subseteq S_4$ satisfies the assumption in Theorem 3.8. Remark 4.6 shows that $V_{\mathbb{Z}\gamma \pm \frac{r}{18}\gamma}$, for $r \in \mathbb{Z}$, $1 \leq r \leq 8$, and $r \not\equiv 0 \pmod{3}$ are twelve irreducible $V_{L_2}^{(\sigma)}$ modules of type two. Theorem 3.5 indicates that the twelve irreducible $V_{L_2}^{(\sigma)}$ modules of type two are twelve irreducible $V_{L_2}^{S_4}$ modules of type two, probably isomorphic under $V_{L_2}^{S_4}$, of type two.

As $V_{L_2}^{D_3} \cong V_{\mathbb{Z}\gamma}^+$ irreducible modules, lemma 4.4 shows that

$$\begin{aligned} V_{\mathbb{Z}\gamma+\frac{1}{18}\gamma} &\cong V_{\mathbb{Z}\gamma-\frac{1}{18}\gamma}, V_{\mathbb{Z}\gamma+\frac{2}{18}\gamma} \cong V_{\mathbb{Z}\gamma-\frac{2}{18}\gamma}, V_{\mathbb{Z}\gamma+\frac{4}{18}\gamma} \cong V_{\mathbb{Z}\gamma-\frac{4}{18}\gamma}, \\ V_{\mathbb{Z}\gamma+\frac{5}{18}\gamma} &\cong V_{\mathbb{Z}\gamma-\frac{5}{18}\gamma}, V_{\mathbb{Z}\gamma+\frac{7}{18}\gamma} \cong V_{\mathbb{Z}\gamma-\frac{7}{18}\gamma}, V_{\mathbb{Z}\gamma+\frac{8}{18}\gamma} \cong V_{\mathbb{Z}\gamma-\frac{8}{18}\gamma}, \end{aligned}$$

Notice that $D_3 \subset S_4$, that is, $V_{L_2}^{S_4}$ is a vertex operator subalgebra of $V_{L_2}^{D_3}$. Thus, as $V_{L_2}^{S_4}$ irreducible modules,

$$\begin{aligned} V_{\mathbb{Z}\gamma+\frac{1}{18}\gamma} &\cong V_{\mathbb{Z}\gamma-\frac{1}{18}\gamma}, V_{\mathbb{Z}\gamma+\frac{2}{18}\gamma} \cong V_{\mathbb{Z}\gamma-\frac{2}{18}\gamma}, V_{\mathbb{Z}\gamma+\frac{4}{18}\gamma} \cong V_{\mathbb{Z}\gamma-\frac{4}{18}\gamma}, \\ V_{\mathbb{Z}\gamma+\frac{5}{18}\gamma} &\cong V_{\mathbb{Z}\gamma-\frac{5}{18}\gamma}, V_{\mathbb{Z}\gamma+\frac{7}{18}\gamma} \cong V_{\mathbb{Z}\gamma-\frac{7}{18}\gamma}, V_{\mathbb{Z}\gamma+\frac{8}{18}\gamma} \cong V_{\mathbb{Z}\gamma-\frac{8}{18}\gamma}, \end{aligned}$$

Lemma 5.6. *Definition of g stable module shows that these twelve irreducible $V_{L_2}^{(\sigma)}$ modules are not ρ stable as and*

$$\begin{aligned} V_{\mathbb{Z}\gamma+\frac{1}{18}\gamma}^\rho &\cong V_{\mathbb{Z}\gamma-\frac{1}{18}\gamma}, V_{\mathbb{Z}\gamma+\frac{2}{18}\gamma}^\rho \cong V_{\mathbb{Z}\gamma-\frac{2}{18}\gamma}, V_{\mathbb{Z}\gamma+\frac{4}{18}\gamma}^\rho \cong V_{\mathbb{Z}\gamma-\frac{4}{18}\gamma}, \\ V_{\mathbb{Z}\gamma+\frac{5}{18}\gamma}^\rho &\cong V_{\mathbb{Z}\gamma-\frac{5}{18}\gamma}, V_{\mathbb{Z}\gamma+\frac{7}{18}\gamma}^\rho \cong V_{\mathbb{Z}\gamma-\frac{7}{18}\gamma}, V_{\mathbb{Z}\gamma+\frac{8}{18}\gamma}^\rho \cong V_{\mathbb{Z}\gamma-\frac{8}{18}\gamma}, \end{aligned}$$

Remark 5.7. Actions of ω on the first level of a module show that those six irreducible $V_{L_2}^{S_4}$ modules are not isomorphic. Theorem 3.6 indicates that those six irreducible $V_{L_2}^{(\sigma)}$ modules exhaust irreducible $V_{L_2}^{S_4}$ modules occurring in irreducible σ^i , for $i = 1, 2$, twisted V_{L_2} modules. So, those six irreducible $V_{L_2}^{S_4}$ modules are exactly the six irreducible $V_{L_2}^{S_4}$ modules occurring in irreducible σ^i , for $i = 1, 2$, twisted V_{L_2} modules.

Proposition 5.8. *By Reference [13], let g be an automorphism of V_{L_2} of order $T \neq 1$. Then there exists some vector $u \in (V_{L_2})_1$, such that $g = e^{2\pi i u(0)}$.*

The action of $\{e^{2\pi i h(0)} | h \in (V_{L_2})_1\}$ on V_{L_2} is isomorphic to $SO(3)$, and on $V_{\mathbb{Z}\frac{\alpha}{2}}$ is isomorphic to $SU(2)$. The action of the group generated by $\{\sigma, \tau_1, \tau_2, \tau_3, \rho\}$ on V_{L_2} is isomorphic to S_4 . Proposition 5.8 shows that $\langle \sigma, \tau_1, \tau_2, \tau_3, \rho \rangle$ is a subgroup of $\{e^{2\pi i h(0)} | h \in (V_{L_2})_1\}$. So, $\langle \sigma, \tau_1, \tau_2, \tau_3, \rho \rangle$ acts on $V_{\mathbb{Z}\frac{\alpha}{2}} = M(1) \otimes \mathbb{C}[\frac{1}{2}\mathbb{Z}\alpha]$. Theorem 3.16 is the action of the group $\langle \sigma, \tau_1, \tau_2, \tau_3, \rho \rangle$ on $V_{\mathbb{Z}\frac{\alpha}{2}}$ is a Schur cover of S_4 , which is isomorphic to $GL(2, 3)$, the general linear group of degree 2 over a field of three elements. Thus, by the quantum Galois theory [19]

$$V_{\mathbb{Z}\frac{\alpha}{2}} \cong \bigoplus_{\chi} V_{\chi} \otimes W_{\chi} \quad (5.1)$$

, where χ runs over all irreducible characters of $GL(2, 3)$. The irreducible representations of the group $GL(2, 3)$ are well known, two 1-dimensional, three 2-dimensional, two 3-dimensional, and one 4-dimensional irreducible representations. Denote them by W_1^i, W_2^j, W_3^k, W_4 , where $i = 0, 1, j = 0, 1, 2, k = 0, 1$. The subindex is the dimensional of

the module and the upper indices distinguish the irreducible modules of the same dimension.

Reference [13] shows that

$$V_{L_2} = (V_{\mathbb{Z}\beta}^+)^0 \otimes U_1^0 \oplus (V_{\mathbb{Z}\beta}^+)^1 \otimes U_1^1 \oplus (V_{\mathbb{Z}\beta}^+)^2 \otimes U_1^2 \oplus V_{\mathbb{Z}\beta}^- \otimes U_3, \quad (5.2)$$

and

$$V_{L_2+\frac{1}{2}\alpha} = V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^0 \otimes U_2^0 \oplus V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^1 \otimes U_2^1 \oplus V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^2 \otimes U_2^2 \quad (5.3)$$

Lemma 5.9. *The following isomorphisms hold,*

$$((V_{\mathbb{Z}\beta}^+)^1)^\rho \cong (V_{\mathbb{Z}\beta}^+)^2,$$

and

$$(V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^1)^\rho \cong V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^2.$$

Proof. Lemma 5.4 shows that, $(V_{\mathbb{Z}\beta}^+)^0$ in equation 5.2 becomes $((V_{\mathbb{Z}\beta}^+)^0)^+$, and $((V_{\mathbb{Z}\beta}^+)^0)^-$ in equation 5.1 for $V_{L_2}^{S_4}$. Likewise, $V_{\mathbb{Z}\beta}^-$ in equation 5.2 becomes $(V_{\mathbb{Z}\beta}^-)^+$, and $(V_{\mathbb{Z}\beta}^-)^-$ in equation 5.1 for $V_{L_2}^{S_4}$. Also, $V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^0$ in Equation 5.3 becomes $(V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^0)^+$, and $(V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^0)^-$ in Equation 5.1 for $V_{L_2}^{S_4}$. At this point, there is a total of six nonisomorphic irreducible $V_{L_2}^{S_4}$ modules in Equation 5.1. Notice that total number of irreducible characters of $GL(2, 3)$ is eight. Quantum Galois theory 5.1 shows that the total number of nonisomorphic irreducible $V_{L_2}^{S_4}$ modules in Equation 5.1 is eight. Theorem 3.9 shows that

$$\text{qdim}_{V_{L_2}^{A_4}}(V_{\mathbb{Z}\beta}^+)^1 = \text{qdim}_{V_{L_2}^{A_4}}(V_{\mathbb{Z}\beta}^+)^2 = 1$$

and

$$\text{qdim}_{V_{L_2}^{A_4}}V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^1 = \text{qdim}_{V_{L_2}^{A_4}}V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^2 = 2$$

This forces $((V_{\mathbb{Z}\beta}^+)^1)^\rho \cong (V_{\mathbb{Z}\beta}^+)^2$, and $(V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^1)^\rho \cong V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^2$. \square

Remark 5.10. Lemma 5.9, Equation 5.1, Equation 5.2, Equation 5.3 show that

$$\begin{aligned} V_{L_2} = & ((V_{\mathbb{Z}\beta}^+)^0)^+ \otimes W_1^0 \oplus ((V_{\mathbb{Z}\beta}^+)^0)^- \otimes W_1^1 \oplus (V_{\mathbb{Z}\beta}^+)^1 \otimes W_2^0 \\ & \oplus (V_{\mathbb{Z}\beta}^-)^+ \otimes W_3^0 \oplus (V_{\mathbb{Z}\beta}^-)^- \otimes W_3^1, \end{aligned} \quad (5.4)$$

and

$$V_{L_2+\frac{1}{2}\alpha} = (V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^0)^+ \otimes W_2^1 \oplus (V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^0)^- \otimes W_2^2 \oplus V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^1 \otimes W_4. \quad (5.5)$$

Lemma 5.11. *The following 8 spaces are irreducible $V_{L_2}^{S_4}$ modules*

$$V_{\mathbb{Z}\gamma+\frac{r}{18}\gamma}, \text{ for } (s \in \mathbb{Z}, 1 \leq r \leq 8, \text{ and } r \not\equiv 0 \pmod{3}),$$

$$(V_{\mathbb{Z}\beta}^+)^1, V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^1.$$

Proof. The desired result follows from Lemma 3.4, Lemma 5.9, and Remark 5.7. \square

Remark 5.12. Lemmas 5.4 and 5.11 provide eighteen irreducible $V_{L_2}^{S_4}$ modules, which exhaust irreducible $V_{L_2}^{S_4}$ modules occurring in irreducible $V_{L_2}^{A_4}$ modules. Lemmas 5.4, 5.6, and 5.9 show that there are exactly five irreducible ρ stable $V_{L_2}^{A_4}$ modules. Thus, by Theorem 3.2, there are exactly five irreducible ρ twisted $V_{L_2}^{A_4}$ modules. So, Lemma 3.3, Theorems 3.5 and 3.6 indicate that there are exactly ten extra irreducible $V_{L_2}^{S_4}$ modules except for the eighteen irreducible $V_{L_2}^{S_4}$ modules occurring in irreducible $V_{L_2}^{A_4}$ modules.

Consider the vertex operator algebra isomorphism

$$V_{L_2}^{S_4} \cong (V_{\mathbb{Z}\beta}^+)^{\langle \sigma, \rho \rangle} \cong (V_{\mathbb{Z}\beta}^+)^{D_3}.$$

Theorem 3.6 and Remark 3.13 show that irreducible $V_{L_2}^{S_4}$ modules come from

- irreducible $V_{\mathbb{Z}\beta}^+$ modules,
- irreducible σ twisted $V_{\mathbb{Z}\beta}^+$ modules,
- irreducible σ^2 twisted $V_{\mathbb{Z}\beta}^+$ modules,
- irreducible ρ twisted $V_{\mathbb{Z}\beta}^+$ modules.

Remark 5.13. Let M be an irreducible $V_{\mathbb{Z}\beta}^+$ module. Then, Theorem 3.5 and Lemma 3.4 shows that M is a $V_{L_2}^{A_4}$ module, and a direct sum of irreducible $V_{L_2}^{A_4}$ modules. So, irreducible $V_{L_2}^{S_4}$ modules from irreducible $V_{\mathbb{Z}\beta}^+$ modules are irreducible $V_{L_2}^{S_4}$ modules from irreducible $V_{L_2}^{A_4}$ modules.

Remark 5.14. Let M be an irreducible σ twisted $V_{\mathbb{Z}\beta}^+$ module. Then, Lemmas 3.3, 3.4 shows that M is a $V_{L_2}^{A_4}$ module, and hence a direct sum of irreducible $V_{L_2}^{A_4}$ modules. So, irreducible $V_{L_2}^{S_4}$ modules from irreducible σ twisted $V_{\mathbb{Z}\beta}^+$ modules are irreducible $V_{L_2}^{S_4}$ modules from irreducible $V_{L_2}^{A_4}$ modules. Likewise, irreducible $V_{L_2}^{S_4}$ modules from irreducible σ^2 twisted $V_{\mathbb{Z}\beta}^+$ modules are irreducible $V_{L_2}^{S_4}$ modules from irreducible $V_{L_2}^{A_4}$ modules. Therefore, those ten extra irreducible $V_{L_2}^{S_4}$ modules in Remark 5.12 are from irreducible ρ twisted $V_{\mathbb{Z}\beta}^+$ modules.

Notice that Lemma 4.4 shows that

$$(V_{\mathbb{Z}\beta}^+)^{\langle \rho \rangle} \cong V_{\mathbb{Z}\zeta}^+ \cong V_{L_2}^{D_4}.$$

Reference [22] shows that the lattice vertex operator algebra $V_{n\mathbb{Z}\alpha}^+$ is generated by ω , J , and $E_{n\alpha}$, where

$$J = (x^1(-1))^4 \mathbf{1} - 2x^1(-3)h(-1)\mathbf{1} + \frac{3}{2}(x^1(-2))^2 \mathbf{1}$$

$$E_{n\alpha} = e^{n\alpha} + e^{-n\alpha}.$$

Definition of ρ shows that

$$\rho(E_{2\alpha}) = -E_{2\alpha},$$

and

$$\rho(E_{4\alpha}) = E_{4\alpha}.$$

Notice that $V_{\mathbb{Z}\beta}^+$ is generated by ω, J , and $E_{2\alpha}$, and $V_{\mathbb{Z}\zeta}^+$ is generated by ω, J , and $E_{4\alpha}$. Thus, $(V_{\mathbb{Z}\beta}^+)^{\langle\rho\rangle}$ and $V_{\mathbb{Z}\zeta}^+$ share the same generators. That is, they are not only isomorphic, but also the same vertex operator algebra.

Lemma 5.15. *As $V_{\mathbb{Z}\zeta}^+$ modules,*

$$V_{\mathbb{Z}\beta}^{T_1,+} \cong (V_{\mathbb{Z}\beta}^{T_2,+})^\rho \cong V_{\mathbb{Z}\zeta}^{T_1,+},$$

and

$$V_{\mathbb{Z}\beta}^{T_2,-} \cong (V_{\mathbb{Z}\beta}^{T_1,-})^\rho \cong V_{\mathbb{Z}\zeta}^{T_1,-}.$$

Proof. Since $\zeta = 2\beta$, $\mathbb{Z}\zeta$ acts as 0 on $\mathbb{Z}\beta/(2\mathbb{Z}\beta)$. Thus, definition of $V_{\mathbb{Z}\beta}^{T_i}$, where $i = 1, 2$, shows that $\mathbb{Z}\zeta$ acts as 1 on both $V_{\mathbb{Z}\beta}^{T_1}$ and $V_{\mathbb{Z}\beta}^{T_2}$. So, as $V_{\mathbb{Z}\zeta}^+ = (V_{\mathbb{Z}\beta}^+)^{\langle\rho\rangle}$ modules,

$$(V_{\mathbb{Z}\beta}^{T_2,+})^\rho \cong V_{\mathbb{Z}\zeta}^{T_1,+}, \quad V_{\mathbb{Z}\beta}^{T_1,+} \cong V_{\mathbb{Z}\zeta}^{T_1,+},$$

and

$$(V_{\mathbb{Z}\beta}^{T_2,-})^\rho \cong V_{\mathbb{Z}\zeta}^{T_1,-}, \quad V_{\mathbb{Z}\beta}^{T_1,-} \cong V_{\mathbb{Z}\zeta}^{T_1,-}.$$

Transitivity of congruence yields the desired results. \square

Lemma 5.16. *The following eight spaces are irreducible $V_{\mathbb{Z}\zeta}^+ = (V_{\mathbb{Z}\beta}^+)^{\langle\rho\rangle}$ modules occurring in irreducible ρ twisted $V_{\mathbb{Z}\beta}^+$ modules.*

$$V_{\mathbb{Z}\zeta + \frac{s}{32}\zeta}, \text{ for } s \in \mathbb{Z}, 1 \leq s \leq 15, \text{ and } s \not\equiv 0 \pmod{2},$$

Proof. Reference [22] shows that these eight spaces are irreducible $V_{\mathbb{Z}\zeta}^+ = (V_{\mathbb{Z}\beta}^+)^{\langle\rho\rangle}$ modules. Let M be an irreducible $V_{\mathbb{Z}\zeta}^+ = (V_{\mathbb{Z}\beta}^+)^{\langle\rho\rangle}$ module from an irreducible $V_{\mathbb{Z}\beta}^+$ module. Then, the action of ω on the first level of M is $\lambda + n$, where λ is the action of ω on the first level of the irreducible $V_{\mathbb{Z}\beta}^+$ module, and n is a nonnegative integer. Check the action of ω on the first level of each module listed in the lemma. None of them satisfies this condition. Theorem 3.6 shows that an irreducible $V_{\mathbb{Z}\zeta}^+ = (V_{\mathbb{Z}\beta}^+)^{\langle\rho\rangle}$ module is from irreducible ρ twisted $V_{\mathbb{Z}\beta}^+$ modules, or from irreducible $V_{\mathbb{Z}\beta}^+$ modules. Since these eight modules are not from irreducible $V_{\mathbb{Z}\beta}^+$ modules, they are from irreducible ρ twisted $V_{\mathbb{Z}\beta}^+$ modules. \square

Remark 5.17. Use Theorem 3.9 to check the quantum dimensions of irreducible $V_{\mathbb{Z}\beta}^+$ modules, and of $V_{\mathbb{Z}\zeta}^+$. The four irreducible $V_{\mathbb{Z}\zeta}^+$ modules, $\{V_{\mathbb{Z}\zeta}^{T_1,\pm}, V_{\mathbb{Z}\zeta}^{T_2,\pm}\}$, are either from $\{V_{\mathbb{Z}\beta}^{T_1,\pm}, V_{\mathbb{Z}\beta}^{T_2,\pm}\}$, or from ρ irreducible twisted $V_{\mathbb{Z}\beta}^+$ modules. Lemma 5.15 shows that $V_{\mathbb{Z}\zeta}^{T_1,\pm}$ are from $\{V_{\mathbb{Z}\beta}^{T_1,\pm}, V_{\mathbb{Z}\beta}^{T_2,\pm}\}$. Hence, $V_{\mathbb{Z}\zeta}^{T_2,\pm}$ are from irreducible ρ twisted $V_{\mathbb{Z}\beta}^+$ modules.

Remark 5.18. Lemma 5.16 and Remark 5.17 provide ten irreducible $V_{\mathbb{Z}\zeta}^+$ occurring in irreducible ρ twisted $V_{\mathbb{Z}\beta}^+$ modules. Let M be an irreducible ρ twisted $V_{\mathbb{Z}\beta}^+$ module. Notice that $C_{D_3}(\rho) = \langle \rho \rangle$, and $\langle \rho \rangle \subset G_M$. The fact $G_M \subset C_{D_3}(\rho)$ shows that $\langle \rho \rangle = G_M$. Theorem 3.5 indicates that the ten irreducible $V_{\mathbb{Z}\zeta}^+$ modules are ten irreducible $V_{L_2}^{S_4}$ modules, probably isomorphic under $V_{L_2}^{S_4}$, occurring in irreducible ρ twisted $V_{\mathbb{Z}\beta}^+$ modules.

Remark 5.19. Actions of ω on the first level of a module show that those ten irreducible $V_{L_2}^{S_4}$ modules in Remark 5.18 are not isomorphic. Theorem 3.6 indicates that those ten irreducible $V_{\mathbb{Z}\zeta}^+$ modules exhaust irreducible $V_{L_2}^{S_4}$ modules occurring in irreducible ρ twisted $V_{\mathbb{Z}\beta}^+$ modules. So, those ten irreducible $V_{L_2}^{S_4}$ modules are the exactly ten irreducible $V_{L_2}^{S_4}$ modules occurring in irreducible ρ twisted $V_{\mathbb{Z}\beta}^+$ modules.

Theorem 5.20. *Irreducible modules of $V_{L_2}^{S_4}$ are*

$$\begin{aligned}
& ((V_{\mathbb{Z}\beta}^+)^0)^+, ((V_{\mathbb{Z}\beta}^+)^0)^-, \\
& (V_{\mathbb{Z}\beta}^-)^+, (V_{\mathbb{Z}\beta}^-)^-, \\
& (V_{\mathbb{Z}\beta+\frac{1}{8}\beta})^+, (V_{\mathbb{Z}\beta+\frac{1}{8}\beta})^-, \\
& (V_{\mathbb{Z}\beta+\frac{3}{8}\beta})^+, (V_{\mathbb{Z}\beta+\frac{3}{8}\beta})^-, \\
& (V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^0)^+, (V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^0)^-, \\
& (V_{\mathbb{Z}\beta}^+)^1, V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^1, \\
& V_{\mathbb{Z}\gamma+\frac{r}{18}\gamma}, \text{ for } r \in \mathbb{Z}, 1 \leq r \leq 8, \text{ and } r \not\equiv 0 \pmod{3}, \\
& V_{\mathbb{Z}\zeta+\frac{s}{32}\zeta}, \text{ for } s \in \mathbb{Z}, 1 \leq s \leq 15, \text{ and } s \not\equiv 0 \pmod{2}, \\
& V_{\mathbb{Z}\zeta}^{T_2,+}, V_{\mathbb{Z}\zeta}^{T_2,-}.
\end{aligned}$$

Proof. The desired result follows from Remark 5.12, Remark 5.14, and Remark 5.19. \square

Theorem 5.21. *The quantum dimensions for all irreducible $V_L^{S_4}$ -modules over $V_L^{S_4}$ are given by the following tables.*

	$((V_{\mathbb{Z}\beta}^+)^0)^+$	$((V_{\mathbb{Z}\beta}^+)^0)^-$	$(V_{\mathbb{Z}\beta}^+)^1$	$(V_{\mathbb{Z}\beta}^-)^+$	$(V_{\mathbb{Z}\beta}^-)^-$
qdim	1	1	2	3	3
	M^0	M^1	M^2	M^3	M^4

	$(V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^0)^+$	$(V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^0)^-$	$V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^1$
qdim	2	2	4
	M^6	M^7	M^8

	$(V_{\mathbb{Z}\beta+\frac{1}{8}\beta})^+$	$(V_{\mathbb{Z}\beta+\frac{1}{8}\beta})^-$	$(V_{\mathbb{Z}\beta+\frac{3}{8}\beta})^+$	$(V_{\mathbb{Z}\beta+\frac{3}{8}\beta})^-$
qdim	6	6	6	6
	M^9	M^{10}	M^{11}	M^{12}

	$V_{\mathbb{Z}\gamma + \frac{r}{18}\gamma}$	$V_{\mathbb{Z}\zeta + \frac{s}{32}\zeta}$	$V_{\mathbb{Z}\zeta}^{T_2,+}$	$V_{\mathbb{Z}\zeta}^{T_2,-}$
qdim	8	6	12	12
	M^{13}, \dots, M^{20}	M^{21}, \dots, M^{26}	M^{27}	M^{28}

In the last table, $r, s \in \mathbb{Z}$, $1 \leq r \leq 8$, $1 \leq s \leq 15$, $r \not\equiv 0 \pmod{3}$, and $s \not\equiv 0 \pmod{2}$.

Proof. The definition of $V_L^{T_i, \pm}$, for $i = 1, 2$, indicates

$$\text{qdim}_{V_L^+} V_L^{T_i, \pm} = \text{qdim}_{V_L^+} V_{\mathbb{Z}\alpha}.$$

Hence, the quantum dimensions listed are obtained by Theorem 3.9 and Theorem 3.10.

□

6 Irreducible modules of $V_{L_2}^{A_5}$

Let $\beta = 2\alpha$, $\gamma = 3\alpha$, and $\mu = 5\alpha$.

Remark 6.1. A collection of properties of the alternating group A_5 are given.

- (a) A_5 is simple.
- (b) Subgroups of A_5 of a fixed order has a unique conjugacy class.
- (c) The maximal proper subgroups of A_5 are isomorphic to S_3 , D_5 , and A_4 .
- (d) The Sylow 2 group is isomorphic to the Klein four group K . The Sylow 3 group is isomorphic to the cyclic group \mathbb{Z}_3 . The Sylow 5 group is isomorphic to the cyclic group \mathbb{Z}_5 .
- (e) The projective special linear group of degree two for A is $PSL(2, 5)$. The corresponding Schur cover is $SL(2, 5)$.

Remark 6.2. Denote the corresponding groups in A by V_4 , $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/5\mathbb{Z}$, twisted S_3 , D_5 , and A_4 . Use $C_G(H)$ and $N_G(H)$ to denote centralizer and normalizer respectively. Then,

$$C_{A_5}(V_4) = V_4, \quad C_{A_5}(\mathbb{Z}/3\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}, \quad C_{A_5}(\mathbb{Z}/5\mathbb{Z}) = \mathbb{Z}/5\mathbb{Z},$$

and

$$N_{A_5}(V_4) = A_4, \quad N_{A_5}(\mathbb{Z}/3\mathbb{Z}) = S_3, \quad N_{A_5}(\mathbb{Z}/5\mathbb{Z}) = D_5,$$

Remark 6.3. There are two irreducible V_{L_2} modules, V_{L_2} and $V_{L_2 + \frac{\alpha}{2}}$. The action of ω on the first level of V_{L_2} is 0, and on $V_{L_2 + \frac{\alpha}{2}}$ is $\frac{1}{4}$. Thus, Definition 3.1 shows that both V_{L_2} and $V_{L_2 + \frac{\alpha}{2}}$ are g stable, for each $g \in A_5$. Let J be a subgroup of A_5 . Remark 3.7 shows that there are two types of irreducible $V_{L_2}^J$ modules.

- An irreducible $V_{L_2}^J$ module M is of *type one* if M occurs in the decomposition of $V_{\frac{\mathbb{Z}\alpha}{2}}$, as $V_{L_2}^J$ modules.

- An irreducible $V_{L_2}^J$ module M is of *type two* if M does not occur in the decomposition of $V_{\frac{\mathbb{Z}\alpha}{2}}$, as $V_{L_2}^J$ modules. That is, M occurs in a h twisted V_{L_2} module for some $h \in J$ and $h \neq 1$.

Remark 6.4. Use Theorem 4.1, Lemma 4.3, Lemma 4.4, and Remark 4.6.

- Irreducible $V_{L_2}^{V_4}$ modules of type two are $V_{\mathbb{Z}\beta+\frac{1}{8}\beta}$, $V_{\mathbb{Z}\beta+\frac{3}{8}\beta}$, $V_{\mathbb{Z}\beta}^{T_1,\pm}$, and $V_{\mathbb{Z}\beta}^{T_2,\pm}$.
- Irreducible $V_{L_2}^{\mathbb{Z}/3\mathbb{Z}}$ modules of type two are $V_{\mathbb{Z}\gamma\pm\frac{r}{18}\gamma}$, for $r \in \mathbb{Z}$, $1 \leq r \leq 17$, and $r \not\equiv 0 \pmod{3}$.
- Irreducible $V_{L_2}^{\mathbb{Z}/5\mathbb{Z}}$ modules of type two are $V_{\mathbb{Z}\mu\pm\frac{r}{18}\gamma}$, for $t \in \mathbb{Z}$, $1 \leq t \leq 49$, and $t \not\equiv 0 \pmod{5}$.

Lemma 6.5. *Let $H \in \{V_4, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}\}$. Then, an irreducible $V_{L_2}^H$ module of type two is an irreducible $V_{L_2}^{A_5}$ module of type two.*

Proof. Notice that $H \subseteq A_5$ satisfies the assumption in Theorem 3.8. The desired result follows from Theorem 3.8. \square

Lemma 6.6. *Let $H \in \{V_4, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}\}$. Then, nonisomorphic irreducible $V_{L_2}^{N_{A_5}(H)}$ modules of type two are nonisomorphic irreducible $V_{L_2}^{A_5}$ modules of type two.*

Proof. Consider $H = V_4$. Then, $N_{A_5}(H) = A_4$. Reference [11] shows that those six irreducible $V_{L_2}^{V_4}$ modules of type two in Remark 6.1 become two nonisomorphic irreducible $V_{L_2}^{A_4}$ modules, $V_{\mathbb{Z}\beta+\frac{1}{8}\beta}$, $V_{\mathbb{Z}\beta+\frac{3}{8}\beta}$. Lemma 6.5 shows that $V_{\mathbb{Z}\beta+\frac{1}{8}\beta}$ and $V_{\mathbb{Z}\beta+\frac{3}{8}\beta}$ are irreducible $V_{L_2}^{A_5}$ modules. The action of ω on the first levels of these two modules are distinct. Thus, they are nonisomorphic irreducible $V_{L_2}^{A_5}$ modules.

Consider $H = \mathbb{Z}/3\mathbb{Z}$. Then, $N_{A_5}(H) = S_3$. Theorem 4.1 shows that those twelve irreducible $V_{L_2}^{\mathbb{Z}/3\mathbb{Z}}$ modules of type two in Remark 6.1 become six nonisomorphic irreducible $V_{L_2}^{S_3}$ modules, $V_{\mathbb{Z}\gamma\pm\frac{r}{18}\gamma}$, for $r \in \mathbb{Z}$, $1 \leq r \leq 8$, and $r \not\equiv 0 \pmod{3}$. Lemma 6.5 shows that those six modules are irreducible $V_{L_2}^{A_5}$ modules. The action of ω on the first levels of these six modules are distinct. Thus, they are nonisomorphic irreducible $V_{L_2}^{A_5}$ modules.

Consider $H = \mathbb{Z}/5\mathbb{Z}$. Then, $N_{A_5}(H) = D_5$. Theorem 4.1 shows that those forty irreducible $V_{L_2}^{\mathbb{Z}/5\mathbb{Z}}$ modules of type two in Remark 6.1 become twenty nonisomorphic irreducible $V_{L_2}^{D_5}$ modules, $V_{\mathbb{Z}\mu\pm\frac{t}{50}\mu}$, for $t \in \mathbb{Z}$, $1 \leq t \leq 24$, and $r \not\equiv 0 \pmod{5}$. Lemma 6.5 shows that those twenty modules are irreducible $V_{L_2}^{A_5}$ modules. The action of ω on the first levels of these twenty modules are distinct. Thus, they are nonisomorphic irreducible $V_{L_2}^{A_5}$ modules. \square

Lemma 6.7. *Irreducible $V_{L_2}^{A_5}$ modules of type two are those twenty eight modules constructed in the proof of Lemma 6.6.*

Proof. The desired result follows from Lemma 6.5, Lemma 6.6, Theorem 3.12, and Remark 6.1. \square

Proposition 5.8 shows that A_5 can be considered as a subgroup of $\{e^{2\pi i h(0)} | h \in (V_{L_2})_1\}$. Thus, A_5 acts on $V_{\mathbb{Z}\frac{\alpha}{2}} = M(1) \otimes \mathbb{C}[\frac{1}{2}\mathbb{Z}\alpha]$. Remark 3.18 shows that the action of the group A_5 on $V_{\mathbb{Z}\frac{\alpha}{2}}$ is a Schur cover of A_4 , which is isomorphic to $SL(2, 5)$, the general linear group of degree 2 over a field of three elements. Thus, by the quantum Galois theory [19]

$$V_{\mathbb{Z}\frac{\alpha}{2}} \cong \bigoplus_{\chi} V_{\chi} \otimes W_{\chi} \quad (6.1)$$

, where χ runs over all irreducible characters of $SL(2, 5)$. The irreducible representations of the group $SL(2, 5)$ are well known, one 1-dimensional, two 2-dimensional, two 3-dimensional, two 4-dimensional, one 5-dimensional, and one 6-dimensional irreducible representations. Denote them by $X_1, X_2^i, X_3^j, X_k, X_5, X_6$, where $i = 0, 1, j = 0, 1, k = 0, 1$. The subindex is the dimension of the module and the upper indices distinguish the irreducible modules of the same dimension.

Remark 6.8. Irreducible $V_{L_2}^{A_5}$ modules of type one occurs and exhausts the modules in the complete decomposition in Equation 6.1. By quantum Galois Theory, express these modules as $T_1, T_2^i, T_3^j, T_4^k, T_5, T_6$, where $i = 0, 1, j = 0, 1, k = 0, 1$. The subindex is the quantum dimension of the module and the upper indices distinguish the irreducible modules of the same dimension.

Theorem 6.9. *Assume that $V_{L_2}^{A_5}$ is rational and C_2 cofiniteness. There are thirty seven irreducible $V_{L_2}^{A_5}$ modules.*

- $T_1, T_2^i, T_3^j, T_4^k, T_5, T_6$, where $i = 0, 1, j = 0, 1, k = 0, 1$,
- $V_{\mathbb{Z}\beta + \frac{1}{8}\beta}, V_{\mathbb{Z}\beta + \frac{3}{8}\beta}$,
- $V_{L_2}^{S_3}$ modules, $V_{\mathbb{Z}\gamma \pm \frac{r}{18}\gamma}$, for $r \in \mathbb{Z}$, $1 \leq r \leq 8$, and $r \not\equiv 0 \pmod{3}$,
- $V_{L_2}^{D_5}$ modules, $V_{\mathbb{Z}\mu \pm \frac{t}{50}\mu}$, for $t \in \mathbb{Z}$, $1 \leq t \leq 24$, and $t \not\equiv 0 \pmod{5}$,

Proof. The desired results follows from Lemma 6.7 and remark 6.8. \square

Theorem 6.10. *The quantum dimensions for all irreducible $V_{L_2}^{A_5}$ -modules over $V_{L_2}^{A_5}$ are given by the following tables.*

	T_1	T_2^i	T_3^j	T_4^k	T_5	T_6
qdim	1	2	3	4	5	6

	$V_{\mathbb{Z}\beta + \frac{1}{8}\beta}$	$V_{\mathbb{Z}\beta + \frac{3}{8}\beta}$	$V_{\mathbb{Z}\gamma \pm \frac{r}{18}\gamma}$	$V_{\mathbb{Z}\mu \pm \frac{t}{50}\mu}$
qdim	30	30	20	12

In the first table, $i = 0, 1, j = 0, 1, k = 0, 1$. In the second table, $r, s \in \mathbb{Z}$, $1 \leq r \leq 8$, $1 \leq s \leq 15$, $r \not\equiv 0 \pmod{3}$, and $s \not\equiv 0 \pmod{2}$.

Proof. The quantum dimensions listed are obtained by Remark 6.8, Theorem 3.9, and Theorem 3.10. \square

Remark 6.11. These classification of $V_{L_2}^{A_5}$ is based on the C_2 cofiniteness and on the rationality of $V_{L_2}^{A_5}$. However, Theorem 3.8 requires the rationality of V^H rather than the rationality of V^G . For any proper subset H of A_5 , notice that V^H is rational. Therefore, even if $V_{L_2}^{A_5}$ was not rational, the irreducible modules listed in this paper would still be correct irreducible modules, but might not exhaust all of them.

References

- [1] Toshiyuki Abe. Fusion rules for the free bosonic orbifold vertex operator algebra. *Journal of Algebra*, 229(1):333–374, 2000.
- [2] Toshiyuki Abe. Rationality of the vertex operator algebra V_L^+ for a positive definite even lattice L . *Mathematische Zeitschrift*, 249(2):455–484, 2005.
- [3] Toshiyuki Abe, Geoffrey Buhl, and Chongying Dong. Rationality, regularity, and C_2 -cofiniteness. *Transactions of the American Mathematical Society*, 356(8):3391–3402, 2004.
- [4] Richard E Borcherds. Vertex algebras, kac-moody algebras, and the monster. *Proceedings of the National Academy of Sciences*, 83(10):3068–3071, 1986.
- [5] C. Dong and C. Jiang. A characterization of vertex operator algebras $V_{\mathbb{Z}\alpha}^+$: I. *ArXiv e-prints*, October 2011.
- [6] C. Dong, L. Ren, and F. Xu. On Orbifold Theory. *ArXiv e-prints*, July 2015.
- [7] Chongying Dong and Robert L Griess. Rank one lattice type vertex operator algebras and their automorphism groups. *Journal of Algebra*, 208(1):262–275, 1998.
- [8] Chongying Dong and Jianzhi Han. On rationality of vertex operator superalgebras. *International Mathematics Research Notices*, 2014(16):4379–4399, 2014.
- [9] Chongying Dong and Cuipo Jiang. A characterization of vertex operator algebra $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$. *Communications in Mathematical Physics*, 296(1):69–88, 2010.
- [10] Chongying Dong and Cuipo Jiang. A characterization of the rational vertex operator algebra: II. *Advances in Mathematics*, 247:41–70, 2013.
- [11] Chongying Dong and Cuipo Jiang. Representations of the vertex operator algebra $V_{L_2}^{A_4}$. *Journal of Algebra*, 377:76 – 96, 2013.
- [12] Chongying Dong and Cuipo Jiang. A characterization of the vertex operator algebra $V_{L_2}^{A_4}$. In *Conformal Field Theory, Automorphic Forms and Related Topics*, pages 55–74. Springer, 2014.

- [13] Chongying Dong, Cuipo (Cuibo) Jiang, Qifen Jiang, Xiangyu Jiao, and Nina Yu. Fusion rules for the vertex operator algebra $V_{L_2}^{A_4}$. *Journal of Algebra*, 423:476 – 505, 2015.
- [14] Chongying Dong, Xiangyu Jiao, and Feng Xu. Quantum dimensions and quantum galois theory. *Transactions of the American Mathematical Society*, 365(12):6441–6469, 2013.
- [15] Chongying Dong, Haisheng Li, and Geoffrey Mason. Regularity of rational vertex operator algebras. *Advances in Mathematics*, 132(1):148–166, 1997.
- [16] Chongying Dong, Haisheng Li, and Geoffrey Mason. Twisted representations of vertex operator algebras. *Mathematische Annalen*, 310(3):571–600, 1998.
- [17] Chongying Dong, Haisheng Li, and Geoffrey Mason. Modular-invariance of trace functions in orbifold theory and generalized moonshine. *Communications in Mathematical Physics*, 214(1):1–56, 2000.
- [18] Chongying Dong, Kefeng Liu, and Xiaonan Ma. Elliptic genus and vertex operator algebras. *Pure and Applied Mathematics Quarterly*, 1(4), 2005.
- [19] Chongying Dong and Geoffrey Mason. On quantum galois theory. *Duke Math. J.*, 86(2):305–321, 1997.
- [20] Chongying Dong and Geoffrey Mason. Rational vertex operator algebras and the effective central charge. *International Mathematics Research Notices*, 2004(56):2989–3008, 2004.
- [21] Chongying Dong and Kiyokazu Nagatomo. Classification of Irreducible Modules for the Vertex Operator Algebra $M(1)^+$. *Journal of Algebra*, 216(1):384 – 404, 1999.
- [22] Chongying Dong and Kiyokazu Nagatomo. Representations of vertex operator algebra V_L^+ for rank one lattice L . *Communications in mathematical physics*, 202(1):169–195, 1999.
- [23] Chongying Dong and Kiyokazu Nagatomo. Classification of irreducible modules for the vertex operator algebra $M(1)^+$: II. higher rank. *Journal of Algebra*, 240(1):289–325, 2001.
- [24] CY Dong. Twisted modules for vertex algebras associated with even lattices. *Journal of Algebra*, 165(1):91–112, 1994.
- [25] Igor Frenkel, Yi-Zhi Huang, and James Lepowsky. *On axiomatic approaches to vertex operator algebras and modules*, volume 494. American Mathematical Soc., 1993.
- [26] Igor Frenkel, James Lepowsky, and Arne Meurman. *Vertex operator algebras and the Monster*, volume 134. Academic press, 1989.

- [27] P Ginsparg. Curiosities at $c = 1$. *Nuclear Physics B*, 295(2):153–170, 1988.
- [28] Akihide Hanaki, Masahiko Miyamoto, Daisuke Tambara, et al. Quantum galois theory for finite groups. *Duke mathematical journal*, 97(3):541–544, 1999.
- [29] P. N. Hoffman and J. F. Humphreys. *Projective representations of the symmetric groups*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1992. *Q*-functions and shifted tableaux, Oxford Science Publications.
- [30] Yi-Zhi Huang. Vertex operator algebras and the verlinde conjecture. *Communications in Contemporary Mathematics*, 10(01):103–154, 2008.
- [31] Elias B Kiritsis. Proof of the completeness of the classification of rational conformal theories with $c = 1$. *Physics Letters B*, 217(4):427–430, 1989.
- [32] M. Miyamoto. Flatness and Semi-Rigidity of Vertex Operator Algebras. *ArXiv e-prints*, April 2011.
- [33] Masahiko Miyamoto. Representation theory of code vertex operator algebra. *Journal of Algebra*, 201(1):115–150, 1998.
- [34] Masahiko Miyamoto. Flatness of tensor products and semi-rigidity for C_2 -cofinite vertex operator algebras i. *arXiv preprint arXiv:0906.1407*, 2009.
- [35] Masahiko Miyamoto. C_2 -cofiniteness of cyclic-orbifold models. *Communications in Mathematical Physics*, 335(3):1279–1286, 2015.
- [36] J Schur. On the representation of the symmetric and alternating groups by fractional linear substitutions. *International Journal of Theoretical Physics*, 40(1):413–458, 2001.
- [37] Erik Verlinde. Fusion rules and modular transformations in 2D conformal field theory. *Nuclear Physics B*, 300:360–376, 1988.
- [38] Robert A. Wilson. *The finite simple groups*. London: Springer, 2009.
- [39] Feng Xu. Algebraic orbifold conformal field theories. *Proceedings of the National Academy of Sciences*, 97(26):14069–14073, 2000.
- [40] Wei Zhang and Chongying Dong. W -algebra $W(2, 2)$ and the vertex operator algebra $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$. *Communications in Mathematical Physics*, 285(3):991–1004, 2009.
- [41] Yongchang Zhu. Modular invariance of characters of vertex operator algebras. *Journal of the American Mathematical Society*, 9(1):237–302, 1996.